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New approach to study the van der Pol equation for large damping

Klaus R. Schneider ✉

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany

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Abstract. We present a new approach to establish the existence of a unique limit cycle for the van der Pol equation in case of large damping. It is connected with the bifurcation of a stable hyperbolic limit cycle from a closed curve composed of two heteroclinic orbits and of two segments of a straight line forming continua of equilibria. The proof is based on a linear time scaling (instead of the nonlinear Liénard transformation in previous approaches), on a Dulac–Cherkas function and on the property of rotating vector fields.

Keywords: relaxation oscillations, Dulac–Cherkas function, rotated vector field.

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1 Introduction

The scalar autonomous differential equation

$$\frac{d^2x}{dt^2} + \lambda(x^2 - 1)\frac{dx}{dt} + x = 0, \tag{1.1}$$

where λ is a real parameter, has been introduced by Balthasar van der Pol [7] in 1926 to describe self-oscillations in a triod circuit.

It is well-known (see e.g. [6]) that for all $\lambda \neq 0$ this equation has a unique isolated periodic solution $p_\lambda(t)$ (limit cycle) whose amplitude stays very near 2 for all λ ([1,9]), but its maximum velocity grows unboundedly as $|\lambda|$ tends to infinity.

If we rewrite the second order equation (1.1) as the planar system

$$\begin{aligned} \frac{dx}{dt} &= -y, \\ \frac{dy}{dt} &= x - \lambda(x^2 - 1)y \end{aligned} \tag{1.2}$$

and if we represent $p_\lambda(t)$ as a closed curve Γ_λ in the (x, y)-phase plane, then Γ_λ does not stay in any bounded region as $|\lambda|$ tends to ∞.

✉ Email: schneider@wias-berlin.de

In the paper [8], van der Pol noticed that for large parameter λ the form of the oscillations "is characterized by discontinuous jumps arising each time the system becomes unstable" and that "the period of these new oscillations is determined by the duration of a capacitance discharge, which is sometimes named a 'relaxation time'." From that reason, van der Pol popularized the name relaxation oscillations [3].

A proof of the conjecture of van der Pol that the limit cycles Γ_λ for large λ is located in a small neighborhood of a closed curve representing a discontinuous period solution has been given by A. D. Flanders and J. J. Stoker [2] in 1946 by considering (1.1) in the so-called Liénard plane [5] which is related to the *nonlinear* Liénard transformation of the phase plane which implies that the corresponding family $\{\Gamma_\lambda\}$ of limit cycles stays for all λ in a uniformly bounded region of the Liénard plane. For defining the Liénard transformation for $\lambda > 0$ we introduce a new time τ by $t = \lambda\tau$ and a new parameter ε by $\varepsilon = \frac{1}{\lambda^2}$. By this way, equation (1.1) can be rewritten in the form

$$\frac{d}{d\tau}\left[\varepsilon \frac{dx}{d\tau} - x + \frac{x^3}{3}\right] + x = 0. \tag{1.3}$$

Applying the nonlinear Liénard transformation

$$\eta = x, \qquad \xi = -\sqrt{\varepsilon} y - x + \frac{x^3}{3} = \varepsilon \frac{dx}{d\tau} - x + \frac{x^3}{3}$$

equation (1.3) is equivalent to the system

$$\begin{aligned} \frac{d\xi}{d\tau} &= -\eta, \\ \varepsilon \frac{d\eta}{d\tau} &= \xi + \eta - \frac{\eta^3}{3} \end{aligned} \tag{1.4}$$

which represents a singularly perturbed system for small $\varepsilon > 0$.

The corresponding degenerate system

$$\begin{aligned} \frac{d\xi}{d\tau} &= -\eta, \\ 0 &= \xi + \eta - \frac{\eta^3}{3} \end{aligned} \tag{1.5}$$

is a differential-algebraic system. This system does not define a dynamical system on the curve $\xi = -\eta + \eta^3/3$ since on that curve there exist two points $A(\eta = -1, \xi = 2/3)$ and $C(\eta = 1, \xi = -2/3)$ which are so-called impass points: no solution of (1.5) can pass these points. Let B (D) be the point where the tangent to the curve $\xi = -\eta + \eta^3/3$ at the point A (C) intersects the curve $\xi = -\eta + \eta^3/3$ (see Figure 1.1). We denote by $\mathcal{Z}_0$ the closed curve in the (η, ξ)-Liénard plane which consists of two segments of the curve $\xi = -\eta + \eta^3/3$ bounded by the points D, A and C, B and of two segments of the straight lines connecting the points A, B and D, C, respectively (see Figure1.1).

Using this closed curve $\mathcal{Z}_0$, D. A. Flanders and J. J. Stoker [2] constructed an annulus whose boundaries are crossed transversally by the trajectories of system (1.4) which contains the stable limit cycle Γ_ε of system (1.4) for sufficiently small positive ε.

In what follows we present a new approach to establish the existence of a unique hyperbolic limit cycle Γ_λ of the van der Pol equation (1.1) for large λ which is based on a linear time scaling instead of applying the mentioned nonlinear Liénard transformation and on using an appropriate Dulac–Cherkas function.

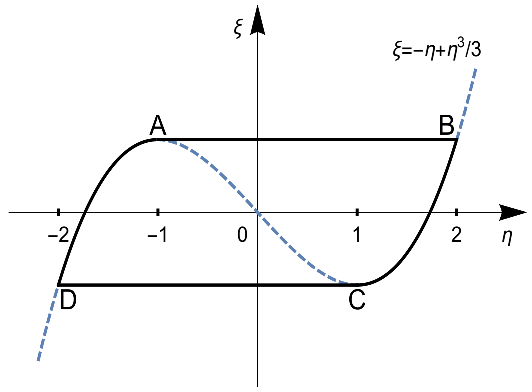

Figure 1.1: Closed curve $\mathcal{Z}_0$.

2 Time scaled van der Pol system

We start with the van der Pol equation (1.1) and apply the time scaling $\sigma = \lambda\ t$ for $\lambda > 0$. Using the notation $\varepsilon = 1/\lambda^2$ we get the equation

$$\frac{d^2x}{d\sigma^2} + (x^2 - 1)\frac{dx}{d\sigma} + \varepsilon x = 0 \tag{2.1}$$

which is equivalent to the system

$$\begin{aligned} \frac{dx}{d\sigma} &= -y, \\ \frac{dy}{d\sigma} &= \varepsilon x - (x^2 - 1)y. \end{aligned} \tag{2.2}$$

Setting $\varepsilon = 0$ in (2.2) we obtain the system

$$\begin{aligned} \frac{dx}{d\sigma} &= -y, \\ \frac{dy}{d\sigma} &= -(x^2 - 1)y \end{aligned} \tag{2.3}$$

which has the x-axis as invariant line consisting of equilibria. In the half-planes $y > 0$ and $y < 0$, the trajectories of system (2.3) are determined by the differential equation

$$\frac{dy}{dx} = x^2 - 1$$

and can be explicitly described by

$$y = \frac{x^3}{3} - x + c, \tag{2.4}$$

where c is any constant.

Next we note the following properties of system (2.2).

Lemma 2.1. *System (2.2) is invariant under the transformation $x \to -x, y \to -y$.*

This lemma implies that the phase portrait of (2.2) is symmetric with respect to the origin.

Lemma 2.2. *The origin is the unique equilibrium of system (2.2), it is unstable for $\varepsilon > 0$.*

Lemma 2.3. *If system (2.2) has a limit cycle Γ_ε, then Γ_ε must intersect the straight lines $x = \pm 1$ in the lower and upper half-planes.*

The proof of Lemma 2.3 is based on the Bendixson criterion and the location of the unique equilibrium.
For the sequel we denote by $X(\varepsilon)$ the vector field defined by (2.2).

Lemma 2.4.
$$\Psi(x, y, \varepsilon) \equiv x^2 + \frac{y^2}{\varepsilon} - 1 \tag{2.5}$$
is a Dulac–Cherkas function of system (2.2) in the phase plane for $\varepsilon > 0$.

Proof. For the function Φ defined in (4.2) in the Appendix, we get from (2.5), (2.2) and $\kappa = -2$
$$\Phi := (\operatorname{grad}\Psi, X(\varepsilon)) + \kappa\Psi \operatorname{div} X(\varepsilon) \equiv 2(x^2-1)^2 \geq 0 \quad \text{for } (x, y, \varepsilon) \in \mathbb{R}^2 \times \mathbb{R}^+. \tag{2.6}$$

Since the set $\mathcal{N}$, where Φ vanishes, consists of the two straight lines $x = \pm 1$ which do not contain a closed curve, we get from Definition 4.1 and Remark 4.2 in the Appendix that Ψ represents a Dulac–Cherkas function for system (2.2) in the phase plane for $\varepsilon > 0$, q.e.d. □

For $\varepsilon > 0$, the set $\mathcal{W}_\varepsilon$, where $\Psi(x, y, \varepsilon)$ vanishes, consists of a unique oval, the ellipse
$$\mathcal{E}_\varepsilon := \left\{(x, y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{\varepsilon} = 1\right\}.$$
We denote by $\mathcal{A}_\varepsilon$ the open simply connected region bounded by $\mathcal{E}_\varepsilon$.

Lemma 2.5. *For $\varepsilon > 0$, $\mathcal{E}_\varepsilon$ is a curve without contact with the trajectories of system (2.1).*

Proof. By Lemma 4.3 in the Appendix, any trajectory of system (2.2) which meets $\mathcal{E}_\varepsilon$ intersects $\mathcal{E}_\varepsilon$ transversally. It is easy to verify that any trajectory of system (2.2) which crosses $\mathcal{E}_\varepsilon$ will leave the region $\mathcal{A}_\varepsilon$ for increasing σ. □

Lemma 2.6. *System (2.2) has no limit cycle in $\mathcal{A}_\varepsilon$ for $\varepsilon > 0$.*

Proof. $\Psi(x, y, \varepsilon)$ is negative for $(x, y) \in \mathcal{A}_\varepsilon$. Thus, by Lemma 4.4 in the Appendix, $|\Psi(x, y, \varepsilon)|^{-2}$ is a Dulac function in the simply connected region $\mathcal{A}_\varepsilon$ for $\varepsilon > 0$ which implies that there is no limit cycle of system (2.2) in $\mathcal{A}_\varepsilon$ for $\varepsilon > 0$. □

Lemma 2.7. *System (2.2) has at most one limit cycle for $\varepsilon > 0$.*

Proof. For $\varepsilon > 0$, the set $\mathcal{W}_\varepsilon$ consists of the unique ellipse $\mathcal{E}_\varepsilon$. Thus, according to Theorem 4.6 in the Appendix, system (2.2) has at most one limit cycle. □

Lemma 2.8. *If system (2.2) has a limit cycle Γ_ε, then Γ_ε is hyperbolic and stable.*

Proof. If system (2.2) has a limit cycle Γ_ε, then by Lemma 2.6 and by Lemma 2.5, Γ_ε must surround the ellipse $\mathcal{E}_\varepsilon$. Thus, we have $\Psi > 0, \kappa = -2, \Phi > 0$ on Γ_ε. Theorem 4.5 in the Appendix completes the proof. □

We recall that for $\lambda > 0$ the van der Pol equation (1.1) and system (2.2) have the same phase portrait. Thus, the results above imply that the van der Pol equation has at most one limit cycle Γ_λ for $\lambda \neq 0$. In the following section we construct for sufficiently small ε a closed curve $\mathcal{K}_\varepsilon$ surrounding the ellipse $\mathcal{E}_\varepsilon$ such that any trajectory of system (2.2) which meets $\mathcal{K}_\varepsilon$ enters for increasing σ the doubly connected region $\mathcal{B}_\varepsilon$ with the boundaries $\mathcal{E}_\varepsilon$ and $\mathcal{K}_\varepsilon$.

3 Existence of a unique limit cycle Γ_ε of system (2.2) for sufficiently small ε

If we assume that there exists some positive number ε_0 such that for $0 < \varepsilon < \varepsilon_0$ system (2.2) has a family $\{\Gamma_\varepsilon\}$ of limit cycles which are uniformly bounded, then the limit position of Γ_ε as ε tends to zero is a bounded closed curve Γ_0 symmetric with respect to the origin which can be described by means of solutions of system (2.3), that is, by segments of curves explicitly given by (2.4) and by segments of the invariant line $y = 0$ consisting of equilibria of system (2.3). Since Γ_0 is a bounded smooth curve in the half-planes $y \geq 0$ and $y \leq 0$ and is symmetric with respect to the origin, we get for Γ_0 in the upper half-plane the representation

$$y = -x + \frac{x^3}{3} + \frac{2}{3} \quad \text{for } -2 \leq x \leq 1, \qquad y = 0 \quad \text{for } 1 \leq x \leq 2.$$

The part in the half-plane $y \leq 0$ is determined by the mentioned symmetry (see Figure 3.1).

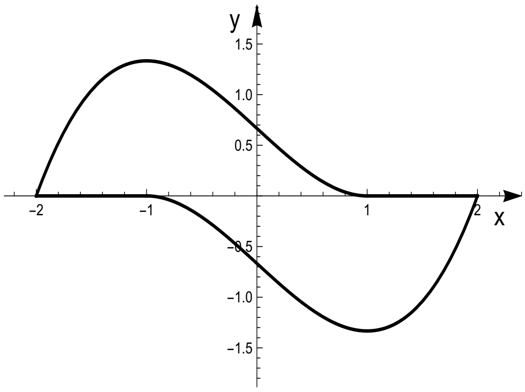

Figure 3.1: Closed curve Γ_0.

We note that Γ_0 consists of two heteroclinic orbits of system (2.3) connecting the equilibria $(-2, 0)$ and $(1, 0)$ in the upper half-plane and the equilibria $(2, 0)$ and $(-1, 0)$ in the lower half-plane, and of the two segments $-2 \leq x \leq -1$ and $1 \leq x \leq 2$ of the x-axis consisting of equilibria of system (2.3).

In what follows, to given $\varepsilon > 0$ sufficiently small, we construct a closed curve $\mathcal{K}_\varepsilon$ surrounding the ellipse $\mathcal{E}_\varepsilon$ such that any trajectory of system (2.2) which meets $\mathcal{K}_\varepsilon$ intersects $\mathcal{K}_\varepsilon$ transversally. For the construction of the closed curve $\mathcal{K}_\varepsilon$ we will take into account the following observations.

Remark 3.1. The amplitude of a possible limit cycle $\{\Gamma_\varepsilon\}$ of system (2.2) tends to 2 as ε tends to zero.

Remark 3.2. Let $\varepsilon > 0$. In the fourth quadrant $(x < 0, y > 0)$, the vector field $X(\varepsilon)$ rotates in the positive sense (anti-clockwise) for increasing ε.

Remark 3.3. Let $\varepsilon > 0$. In the first quadrant, the vector field $X(\varepsilon)$ rotates in the positive sense for decreasing ε.

Let P be the point $(-2.2, 0)$ on the x-axis and let R be the point $(2.2, 0)$ symmetric to P with respect to the origin. By Remark 3.1, there exists a sufficiently small positive number ε_0

such that for $\varepsilon \in (0, \varepsilon_0)$ any limit cycle Γ_ε of system (2.2) intersects the x-axis in the intervals $(-2.2, 0)$ and $(0, 2.2)$. We denote by $\mathcal{T}_0(P)$ the trajectory of system (2.3) in the upper half-plane having P as ω-limit point. $\mathcal{T}_0(P)$ intersects the y-axis at the point $Q_0 = (0, (2.2)^3/3 - 2.2) \approx (0, 1.349)$ and the straight line $x = 2.2$ at the point $R_0 = (2.2, 2[(2.2)^3/3 - 2.2]) \approx (2.2, 2.699)$. $\mathcal{T}_0(P)$ is represented as solid curve in Figure 3.2. Let $\mathcal{T}_{\varepsilon/2}(P)$ be the trajectory of system (2.2), where ε is replaced by $\varepsilon/2$, in the fourth quadrant which passes the point P. For sufficiently small ε, it cuts the y-axis at the point $Q_{\varepsilon/2}$ which is near the point Q_0. $\mathcal{T}_{\varepsilon/2}(P)$ is represented as dotted curve in Figure 3.2.

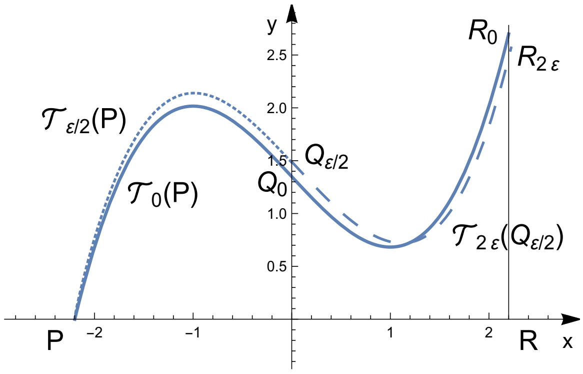

Figure 3.2: Trajectories $\mathcal{T}_0(P), \mathcal{T}_{\varepsilon/2}(P)$ and $\mathcal{T}_{2\varepsilon}(Q_{\varepsilon/2})$ for $\varepsilon = 0.08$.

Now we denote by $\mathcal{T}_{2\varepsilon}(Q_{\varepsilon/2})$ the trajectory of system (2.2), where ε is replaced by 2ε, in the first quadrant which passes the point $Q_{\varepsilon/2}$. For sufficiently small ε, $\mathcal{T}_{2\varepsilon}(Q_{\varepsilon/2})$ is located near the trajectory $\mathcal{T}_0(P)$ and intersects the straight line $x = 2.2$ at the point $R_{2\varepsilon}$ which is located near the point R_0. $\mathcal{T}_{2\varepsilon}(Q_{\varepsilon/2})$ is represented as dashed curve in Figure 3.2.

According to Remark 3.2, any trajectory of system (2.2) (with the parameter ε), which meets the trajectory $\mathcal{T}_{\varepsilon/2}(P)$ in the fourth quadrant, intersects $\mathcal{T}_{\varepsilon/2}(P)$ transversally and enters for increasing σ the region $\tilde{\mathcal{D}}_{\varepsilon/2}$ in the fourth quadrant bounded by $\mathcal{T}_{\varepsilon/2}(P)$, the x-axis and the y-axis (see Figure 3.3). By Remark 3.3, any trajectory of system (2.2) (with the parameter ε) which meets the trajectory $\mathcal{T}_{2\varepsilon}(Q_{\varepsilon/2})$ in the first quadrant intersects $\mathcal{T}_{2\varepsilon}(Q_{\varepsilon/2})$ transversally and enters for increasing σ the region $\tilde{\mathcal{D}}_{2\varepsilon}$ in the fist quadrant bounded by $\mathcal{T}_{2\varepsilon}(Q_{\varepsilon/2})$, the x-axis, the y-axis and the straight line $x = 2.2$ (see Figure 3.3).

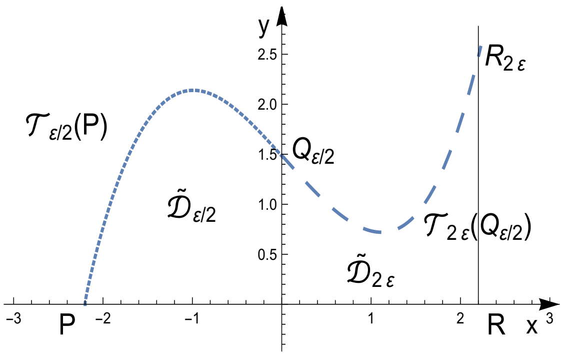

Figure 3.3: Trajectories $\mathcal{T}_{\varepsilon/2}(P)$ and $\mathcal{T}_{2\varepsilon}(Q_{\varepsilon/2})$ for $\varepsilon = 0.08$.

It can be easily verified that the trajectories of system (2.2) intersect the segment of the straight line $x = 2.2$ bounded by the points R and $R_{2\varepsilon}$ transversally in the upper half-plane entering $\tilde{\mathcal{D}}_{2\varepsilon}$ for increasing σ. Since at the point R we have $dy/d\sigma > 0$, also the trajectory of system (2.2) passing R enters the region $\tilde{\mathcal{D}}_{2\varepsilon}$ for increasing σ. If we denote by $\tilde{\mathcal{K}}_{\varepsilon,u}$ the curve composed by the trajectories $\mathcal{T}_{\varepsilon/2}(P)$ and $\mathcal{T}_{2\varepsilon}(Q_{\varepsilon/2})$ and the segment of the straight line $x = 2.2$ bounded by the points R and $R_{2\varepsilon}$, and introduce the set $\tilde{\mathcal{D}}_{\varepsilon,u} = \tilde{\mathcal{D}}_{\varepsilon/2} \cup \tilde{\mathcal{D}}_{2\varepsilon}$, then we have the following result: For sufficiently small ε, any trajectory of system (2.2) which meets the curve $\tilde{\mathcal{K}}_{\varepsilon,u}$ enters for increasing σ the region $\tilde{\mathcal{D}}_{\varepsilon,u}$. Exploiting the symmetry described in Lemma 2.1 there exists a curve $\tilde{\mathcal{K}}_{\varepsilon,l}$ in the lower half-plane such that the closed curve $\tilde{\mathcal{K}}_{\varepsilon} = \tilde{\mathcal{K}}_{\varepsilon,u} \cup \tilde{\mathcal{K}}_{\varepsilon,l}$ bounding the simply connecting region $\tilde{\mathcal{D}}_{\varepsilon}$ has the properties of the wanted closed curve $\mathcal{K}_{\varepsilon}$. Figure 3.4 shows the closed curves $\tilde{\mathcal{K}}_{\varepsilon}$ and $\mathcal{E}_{\varepsilon}$ bounding the annulus $\tilde{\mathcal{B}}_{\varepsilon} = \tilde{\mathcal{D}}_{\varepsilon} \setminus \mathcal{A}_{\varepsilon}$ containing the limit cycle Γ_{ε} for $\varepsilon = 0.08$.

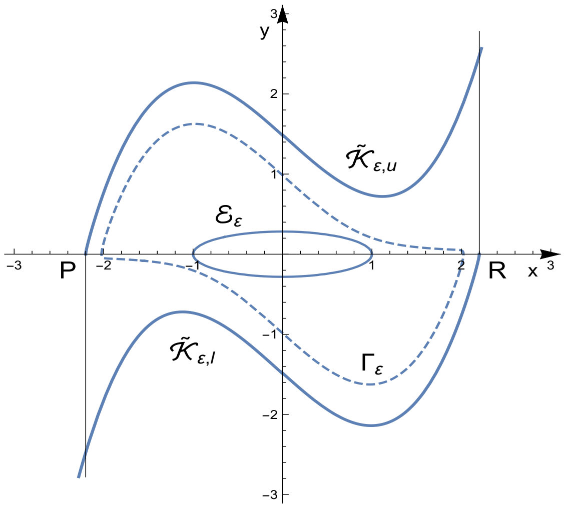

Figure 3.4: Closed curves $\tilde{\mathcal{K}}_{\varepsilon}$ and $\mathcal{E}_{\varepsilon}$ together with the limit cycle Γ_{ε} for $\varepsilon = 0.08$.

An improvement of the curve $\tilde{\mathcal{K}}_{\varepsilon}$ and consequently of the annulus $\tilde{\mathcal{B}}_{\varepsilon}$ containing the limit cycle Γ_{ε} can be obtained as follows. Let $\mathcal{T}_{2\varepsilon}(R)$ be the trajectory of system (2.2), where ε is replaced by 2ε, in the first quadrant starting at the point R. Since on the interval $0 < x < 2.2$ of the x-axis it holds $dy/dx = \varepsilon x > 0$, the trajectory $\mathcal{T}_{2\varepsilon}(R)$ will intersect the y-axis at the point $Q_{2\varepsilon}$ which is located between the origin and the point $Q_{\varepsilon/2}$. We denote by $\mathcal{D}_{2\varepsilon}$ the region in the first quadrant bounded by the trajectory $\mathcal{T}_{2\varepsilon}(R)$, the x-axis and the y-axis (see Figure 3.5). Using Remark 3.3, we can conclude as above that any trajectory of system (2.2) which meets $\mathcal{T}_{2\varepsilon}(R)$ enters $\mathcal{D}_{2\varepsilon}$ for increasing σ. For the following we denote by $\mathcal{I}_{\varepsilon}$ the interval on the y-axis bounded by the points $Q_{\varepsilon/2}$ and $Q_{2\varepsilon}$.

From (2.2) it follows that any trajectory of system (2.2) which meets $\mathcal{I}_{\varepsilon}$ enters for increasing σ the region $\tilde{\mathcal{D}}_{\varepsilon/2}$. Thus, we have the following result: There is a sufficiently small ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$ the trajectories $\mathcal{T}_{\varepsilon/2}(P)$ and $\mathcal{T}_{2\varepsilon}(R)$ and the interval $\mathcal{I}_{\varepsilon}$ form a curve $\mathcal{K}_{\varepsilon,u}$ which bounds in the upper half-plane the bounded region $\mathcal{D}_{\varepsilon,u} = \tilde{\mathcal{D}}_{\varepsilon/2} \cup \mathcal{D}_{2\varepsilon}$ such that any trajectory of system (2.2) which meets $\mathcal{K}_{\varepsilon,u}$ enters for increasing σ the region $\mathcal{D}_{\varepsilon,u}$. Exploiting the symmetry described in Lemma 2.1 there exists a curve $\mathcal{K}_{\varepsilon,l}$ in the lower half-plane such that $\mathcal{K}_{\varepsilon} = \mathcal{K}_{\varepsilon,u} \cup \mathcal{K}_{\varepsilon,l}$ forms the wanted closed curve $\mathcal{K}_{\varepsilon}$. In Figure 3.6 the closed curves $\mathcal{K}_{\varepsilon}$ and $\mathcal{E}_{\varepsilon}$ are represented as solid curves which form the boundaries of the doubly connected region $\mathcal{B}_{\varepsilon}$ which contains the unique limit cycle Γ_{ε} (dashed curve) for $\varepsilon = 0.08$.

Summarizing our considerations we have the following results.

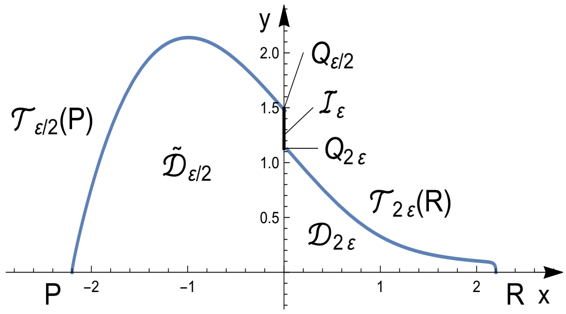

Figure 3.5: Curve $\mathcal{K}_{\varepsilon,u} = \mathcal{T}_{2\varepsilon}(R) \cup \mathcal{I}_\varepsilon \cup \mathcal{T}_{\varepsilon/2(P)}(R)$ for $\varepsilon = 0.08$.

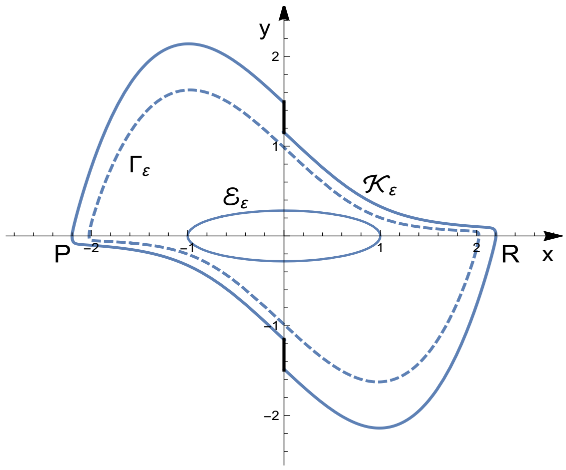

Figure 3.6: Closed curves $\mathcal{K}_\varepsilon$ and $\mathcal{E}_\varepsilon$ and the limit cycle Γ_ε for $\varepsilon = 0.08$.

Theorem 3.4. *For sufficiently small positive ε, the closed curve $\mathcal{K}_\varepsilon$ is a curve without contact for system (2.2), that is, any trajectory of system (2.2) which meets the curve $\mathcal{K}_\varepsilon$ crosses it transversally by entering the doubly connected region $\mathcal{B}_\varepsilon$ for increasing σ.*

Taking into account Lemma 2.7 and Lemma 2.8 we get the following theorem from Theorem 3.4.

Theorem 3.5. *For sufficiently small positive ε, system (2.2) has a unique limit cycle Γ_ε. This cycle is hyperbolic and stable and located in the region $\mathcal{B}_\varepsilon$.*

4 Appendix

Consider the planar autonomous differential system

$$\frac{dx}{dt} = P(x,y,\lambda), \qquad \frac{dy}{dt} = Q(x,y,\lambda) \tag{4.1}$$

depending on a scalar parameter λ. Let Ω be some region in $\mathbb{R}^2$, let Λ be some interval and let $X(\lambda)$ be the vector field defined by system (4.1). Suppose that $P, Q : \Omega \times \Lambda \to \mathbb{R}$ are continuous in all variables and continuously differentiable in the first two variables.

Definition 4.1. A function $\Psi : \Omega \times \Lambda \to \mathbb{R}$ with the same smoothness as P, Q is called a Dulac–Cherkas function of system (4.1) in Ω for $\lambda \in \Lambda$ if there exists a real number $\kappa \neq 0$ such that

$$\Phi := (\operatorname{grad} \Psi, X(\lambda)) + \kappa \Psi \operatorname{div} X(\lambda) > 0 \ (< 0) \quad \text{for } (x, y, \lambda) \in \Omega \times \Lambda. \tag{4.2}$$

Remark 4.2. Condition (4.2) can be relaxed by assuming that Φ may vanish in Ω on a set $\mathcal{N}$ of measure zero, and that no closed curve of $\mathcal{N}$ is a limit cycle of (4.1).

In the theory of Dulac–Cherkas functions the set

$$\mathcal{W}_\lambda := \{(x, y) \in \Omega : \Psi(x, y, \lambda) = 0\} \tag{4.3}$$

plays an important role. The following results can be found in [4].

Lemma 4.3. *Any trajectory of system (4.1) which meets $\mathcal{W}_\lambda$ intersects $\mathcal{W}_\lambda$ transversally.*

Lemma 4.4. *Let $\Omega_1 \subset \Omega$, $\Lambda_1 \subset \Lambda$ be such that $\Psi(x, y, \lambda)$ is different from zero for $(x, y, \lambda) \in \Omega_1 \times \Lambda_1$. Then $|\Psi|^\kappa$ is a Dulac function in Ω_1 for $\lambda \in \Lambda$.*

Theorem 4.5. *Let Ψ be a Dulac–Cherkas function of (4.1) in Ω for $\lambda \in \Lambda$. Then any limit cycle Γ_λ of (4.1) in Ω is hyperbolic and its stability is determined by the sign of the expression $\kappa \Phi \Psi$ on Γ_λ, it is stable (unstable) if $\kappa \Phi \Psi$ is negative (positive) on Γ_λ*

Theorem 4.6. *Let Ω be a p-connected region, let Ψ be a Dulac–Cherkas function of (4.1) in Ω such that the set $\mathcal{W}_\lambda$ consists of s ovals in Ω. Then system (4.1) has at most $p - 1 + s$ limit cycles in Ω.*

References

[1] E. Fisher, The period and amplitude of the Van Der Pol limit cycle, *J. Appl. Phys.* **25**(1954), 273–274. MR0059436

[2] D. A. Flanders, J. J. Stoker, The limit case of relaxation oscillations, in: *Studies in Nonlinear Vibration Theory*, Institute for Mathematics and Mechanics, New York University, 1946, pp. 51–64. MR0018780

[3] J.-M. Ginoux, C. Letellier, Van der Pol and the history of relaxation oscillations: toward the emergence of a concept, *Chaos* **22**(2012), 023120, 15 pp. MR3388537; https://doi.org/10.1063/1.3670008

[4] A. A. Grin, K. R. Schneider, On some classes of limit cycles of planar dynamical systems, *Dyn. Contin. Discrete Impuls. Syst. Ser. A, Math. Anal.* **14**(2007), 641–656. MR2354839

[5] A. Liénard, *Étude des oscillations entretenues* (in French), Revue générale de l'électricité **23**(1928), 906–954.

[6] L. Perko, *Differential equations and dynamical systems*, Texts in Applied Mathematics, Vol. 7, Springer, 2001. https://doi.org/10.1007/978-1-4613-0003-8

[7] B. van der Pol, *On relaxation-oscillations*, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Ser. 7* **2**(1926), 978–992. https://doi.org/10.1080/14786442608564127

[8] B. van der Pol, J. van der Mark, Le battement du cœur considéré comme oscillation de relaxation et un modèle électrique du cœur, *L'Onde Électrique* **7**(1928), 365–392.

[9] H. Yanagiwara, Maximum of the amplitude of the periodic solution of van der Pol's equation, *J. Sci. Hiroshima Univ. Ser. A-I Math.* **25**(1961), 127–134. MR0147709